CLASSIFICATION OF RADIAL SOLUTIONS TO LIOUVILLE SYSTEMS WITH SINGULARITIES

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ABSTRACT. Let $A = (a_{ij})_{n \times n}$ be a nonnegative, symmetric, irreducible and invertible matrix. We prove the existence and uniqueness of radial solutions to the following Liouville system with singularity:

$$\begin{cases} \Delta u_i + \sum_{j=1}^n a_{ij} |x|^{\beta_j} e^{u_j(x)} = 0, & \mathbb{R}^2, \quad i = 1, ..., n \\ \\ \int_{\mathbb{R}^2} |x|^{\beta_i} e^{u_i(x)} dx < \infty, & i = 1, ..., n \end{cases}$$

where $\beta_1, ..., \beta_n$ are constants greater than -2. If all β_is are negative we prove that all solutions are radial and the linearized system is non-degenerate.

1. INTRODUCTION

In this article we consider the following singular Liouville system

(1.1)
$$\begin{cases} \Delta u_i + \sum_{j=1}^n a_{ij} |x|^{\beta_j} e^{u_j(x)} = 0, & \mathbb{R}^2, \quad i \in I := \{1, ..., n\}, \\ \\ \int_{\mathbb{R}^2} |x|^{\beta_i} e^{u_i(x)} dx < \infty, & i \in I. \end{cases}$$

where $\beta_1, .., \beta_n$ are constants greater than -2, $A = (a_{ij})_{n \times n}$ is a constant matrix that satisfies

$(H1):$ A is symmetric, nonnegative, irreducible and invertible.

A is irreducible means there is no disjoint partition of I into I_1 and I_2 such that $a_{ij} = 0$ for all $i \in I_1$ and $j \in I_2$. For the system (1.1), the irreducibility of A means (1.1) can not be written as two independent subsystems. If $n = 1$ and $a_{11} = 1$, the singular Liouville system is reduced to the following single Liouville equation:

(1.2)
$$\Delta u + |x|^{\beta} e^u = 0, \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{\beta} e^u < \infty.$$

Prajapat-Tarantello [46] classified all the solutions to (1.2) and proved, on one hand, that if $\beta/2 \notin \mathbb{N}$, all solutions are radial and can be written as

$$u(x) = \log \frac{\mu}{(1 + \frac{\mu}{8(\beta/2+1)^2} |x|^{\beta+2})^2}, \quad \mu > 0.$$

On the other hand, a solution may not be symmetric around any point if $\beta/2 \in \mathbb{N}$. The proof of Prajapat-Tarantello uses properties of integrable system. However, the Liouville system is not integrable and we have to apply new methods. The purpose of this paper is to prove a classification theorem for all the **radial** solutions to (1.1). Let $u = (u_1, ..., u_n)$ be a solution to (1.1) we use $\sigma = (\sigma_1, ..., \sigma_n)$ to denote its energy:

$$(1.3) \qquad \sigma_i = \frac{1}{2\pi} \int_0^\infty |x|^{\beta_i} e^{u_i(x)} dx, \quad i \in I := \{1, ..., n\}$$

and we set $\Lambda_I(\sigma)$ as

$$\Lambda_I(\sigma) = 2 \sum_{i \in I} (\beta_i + 2)\sigma_i - \sum_{i,j \in I} a_{ij}\sigma_i\sigma_j.$$

For $J \subset I$, $\Lambda_J(\sigma)$ is understood similarly. The main theorem of this article is

Theorem 1.1. *Let A satisfy $(H1)$, $\beta_1, ..., \beta_n > -2$ be constants,*

    (1) *If $u = (u_1, ..., u_n)$ is a radial solution to (1.1), then*

$$(1.4) \qquad \Lambda_I(\sigma) = 0, \ \Lambda_J(\sigma) > 0 \ \forall \emptyset \subsetneq J \subsetneq I.$$

    (2) *For each $\sigma = (\sigma_1, ..., \sigma_n)$ satisfying (1.4), there exists a global radial solution u whose energy is σ.*

    (3) *If u and v are both radial solutions to (1.1) with*

$$\int_0^\infty r^{\beta_i+1} e^{u_i(r)} dr = \int_0^\infty r^{\beta_i+1} e^{v_i(r)} dr, \quad i \in I.$$

    Then $u_i(r) = v_i(\delta r) + (2 + \beta_i)\log\delta$ for some $\delta > 0$ and all $i \in I$.

System (1.1) is reduced to the following form if $\beta_1 = ... = \beta_n = 0$,

$$(1.5) \qquad \begin{cases} \Delta u_i + \sum_{j \in I} a_{ij} e^{u_j} = 0, & \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_i} < \infty, & \mathbb{R}^2. \end{cases}$$

Under the assumption $(H1)$ on A, a standard moving-plane argument shows that all $u_1, .., u_n$ are radially symmetric with respect to a common point (see [22] for the proof). The classification of all solutions to (1.5) has been completed through the works of Chipot-Shafrir-Wolansky [22, 23] and the authors [40]. Among other things Chipot et. al. prove that

Theorem A: (Chipot-Shafrir-Wolansky) Suppose A satisfies $(H1)$, for any solution $u = (u_1, ..., u_n)$ to (1.5), its energy $\sigma = (\sigma_1, ..., \sigma_n)$ belongs to the hypersurface

$$\Gamma := \{\sigma = (\sigma_1, ..., \sigma_n); \ \Lambda_J(\sigma) > 0, \ \Lambda_J(\sigma) > 0, \ \forall \emptyset \subsetneq J \subsetneq I. \}$$

On the other hand, for any $\sigma \in \Gamma$, there is a solution u of (1.5) whose energy is σ.

It can be readily verified that the energy of a solution of (1.5) is invariant under rigid translations and appropriate scalings: Let u be a global solution to (1.5), then $v = (v_1, ..., v_n)$ defined by

$$(1.6) \qquad\qquad v_i(y) = u_i(\delta y + x_0) + 2\log\delta, \quad i \in I$$

for any $x_0 \in \mathbb{R}^2$ and any $\delta > 0$ clearly satisfies $\int_{\mathbb{R}^2} e^{v_i} = \int_{\mathbb{R}^2} e^{u_i}$ for all $i \in I$. It turns out that for any $\sigma \in \Gamma$, all the global solutions that have the energy σ are related by a translation and a scaling described in (1.6):

Theorem B: ([40]) Suppose A satisfies $(H1)$. Let $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$ be global solutions to (1.5) such that $\int_{\mathbb{R}^2} e^{u_i} = \int_{\mathbb{R}^2} e^{v_i}$ for all $i \in I$, then v and u are related by (1.6) for some $\delta > 0$ and $x_0 \in \mathbb{R}^2$.

Theorem A and Theorem B together give a classification of all the solutions to (1.5). One obvious question that Theorem 1.1 raises is, for what $\beta_1, .., \beta_n$ do all the solutions to (1.1) have to be radially symmetric? We give an affirmative answer for the case of non-positive β.

Theorem 1.2. *Let u be a solution to (1.1), A satisfy $(H1)$. Suppose $\beta_i \in (-2, 0]$ for $i \in I$ and are not all equal to 0. Then all components of u are radial functions.*

Systems (1.1) and (1.5) and their reductions appear in many disciplines of mathematics and have profound background in Physics, Chemistry and Ecology. When (1.5) is reduced to one equation, it becomes the classical Liouville equation

$$-\Delta u = e^u,$$

which is related to finding a metric with constant Gauss curvature. In Physics, the Liouville equation represents the electric potential induced by the charge carrier in electrolytes theory [48] and is closely related to the abelian model in the Chern-Simons theories [29, 30, 28].

The Liouville systems (1.5)(1.1) are used to describe models in the theory of chemotaxis [21, 32], in the physics of charged particle beams [5, 24, 33], and in the theory of semi-conductors [42]. For applications of Liouville systems, see [11, 22, 40, 41] and the references therein. Here we note that Liouville systems with singularities are of special importance in Physics and Geometry. For example, the single equation (1.2) appeared in [44] as a limiting equation in the blow-up analysis of periodic vortices for the Chern-Simons theory of Jackiw and Weinberg [27] and Hong et. al. [26]. In geometry (1.1) is related to finding metric with conic singularities [12, 13, 14, 31].

It is well known that classification theorems are closely related to blowup analysis and degree-counting theorems. For many equations the asymptotic behavior of blowup solutions are approximated by global solutions. For example, for the Liouville equation

$$\Delta u + V e^u = 0, \quad \Omega \subset \mathbb{R}^2,$$

if V is a positive smooth function, blowup solutions near a blowup point can be well approximated by global solutions to

$$\Delta u + e^u = 0, \quad \mathbb{R}^2$$

see [19, 34, 15, 57]. If V is nonnegative and the blowup point happens to be a zero of V, the profile of blowup solutions is similar to that of the global solutions of (1.2), see [46, 4, 58]. We expect Theorem 1.1 to be useful in the study of singular Liouville systems defined on Riemann surfaces or domains in $\mathbb{R}^2$.

The proof of the uniqueness part of Theorem 1.1 (the third statement) is motivated by the authors' previous work [40] on the Liouville system with no singularity. The existence part (the second statement) is based on the uniqueness result and is therefore significantly different from the duality method used by Chipot. et. al. in [22]. The first statement in Theorem 1.1 is similar to the corresponding case in [22].

For many applications, especially on the construction of bubbling solutions it is important to study the nondegeneracy of the linearized system. Our next result is concerned with the case when β is non-positive.

Theorem 1.3. *Let $\beta_i \in (-2,0)$ for all $i \in I$, $u = (u_1,...,u_n)$ solve (1.1) corresponding to $\beta = (\beta_1,..,\beta_n)$. Let $\phi = (\phi_1,..,\phi_n)$ be a bounded solution to*

$$\Delta \phi_i + \sum_{j \in I} a_{ij}|y|^{\beta_j} e^{u_j(y)} \phi_j(y) = 0, \quad \mathbb{R}^2, i \in I.$$

Then there exists $C \in \mathbb{R}$ such that $\phi_i(r) = C(ru_i'(r) + 2 + \beta_i)$ for all $i \in I$.

Remark 1.1. *By Theorem 1.2 u is radial in Theorem 1.3.*

The organization of this paper is as follows. In section two we list standard tools to be used in the proof of Theorem 1.1. Then in section three we prove the three statements of Theorem 1.1. Theorem 1.2 and Theorem 1.3 are proved in section four and section five, respectively. Finally in the appendix we provide proofs for the tools used in the proof of Theorem 1.1.

2. PRELIMINARY RESULTS

In this section we list a few ODE lemmas to be used in the proof of Theorem 1.1. Since these lemmas are standard we put their proofs in the appendix, in order not to disturb the main part of the paper.

Lemma 2.1. *Let $u = (u_1,...,u_n)$ be a solution to (1.1) where A satisfies (H1). Then*

$$u_i(x) = -m_i \log|x| + c_i + o(|x|^{-\delta}), \quad i \in I, \quad |x| > 1,$$

$$\nabla u_i(x) = -m_i x/|x|^2 + O(|x|^{-\delta-1}), \quad i \in I, \quad |x| > 1.$$

where

$$m_i = \sum_{j=1}^{n} a_{ij}\sigma_j > 2 + \beta_i, \quad i \in I,$$

$$c_i = u_i(0) + \int_0^\infty \log r \sum_{j=1}^{n} a_{ij} r^{\beta_j+1} e^{u_j(r)} dr$$

δ *is some positive small number.*

Remark 2.1. *u is not assumed to be radial in Lemma 2.1.*

The next lemma is on the linearized system of (1.1) expanded along a radial solution u:

$$(2.1) \qquad (r\phi_i')' + \sum_j a_{ij} r^{\beta_j} e^{u_j} r\phi_j = 0, \quad i \in I.$$

Lemma 2.2. *Let $\phi = (\phi_1, ..., \phi_n)$ satisfy (2.1) with $\beta_i > -2$ for all $i \in I$, then $\phi_i(r) = O(\log r)$ at infinity for $i \in I$.*

Lemma 2.3. *Let A satisfy (H1), $\beta_i > -2$ for $i \in I$, then for any $c_1, ..., c_n \in \mathbb{R}$, there is a unique solution to*

$$(2.2) \qquad \begin{cases} u_i''(r) + \frac{1}{r}u_i'(r) + \sum_{j=1}^{n} a_{ij} r^{\beta_j} e^{u_j(r)} = 0, \quad i = 1, .., n, \\ u_i(0) = c_i, \quad i = 1, .., n \end{cases}$$

that exists for all $r > 0$.

Remark 2.2. *u may not have finite energy.*

If we further know that $a_{ii} > 0$ for all i, then the solution has a finite energy:

Lemma 2.4. *Let $a_{ii} > 0$ for all $i \in I$, then for all $c_1, ..., c_n \in \mathbb{R}$, there exists a solution to*

$$(2.3) \qquad \begin{cases} u_i'' + \frac{1}{r}u_i'(r) + \sum_{j \in I} a_{ij} r^{\beta_j} e^{u_j(r)} = 0, \quad 0 < r < \infty, \quad i \in I, \\ \\ \int_0^\infty e^{u_i(r)} r^{\beta_j+1} dr < \infty, \quad i \in I, \\ \\ u_i(0) = c_i, \quad i \in I. \end{cases}$$

Lemma 2.5. *Let ϕ be a solution of*

$$\begin{cases} (r\phi_i'(r))' + \sum_{j=1}^{n} a_{ij} r^{\beta_j+1} e^{u_j} \phi_j(r) = 0, \quad 0 < r < \infty, \\ \\ \phi_i(0) = 0, \quad \forall i \in I. \end{cases}$$

Suppose $\beta_i > -2$ for all $i \in I$, then $\phi_i \equiv 0$ for all $i \in I$.

3. Proof of Theorem 1.1:

3.1. The proof of the first statement of Theorem 1.1.

Lemma 3.1. *Let $u = (u_1, .., u_n)$ be a radial solution of (1.1) with A satisfying $(H1)$ and $\beta_i > -2$ for all i. Then*

$$\Lambda_I(\sigma) = 0, \quad \Lambda_J(\sigma) > 0, \quad \forall \emptyset \subsetneq J \subsetneq I.$$

Proof of Lemma 3.1: This proof uses the same idea as in [22]. Let $\tilde{u}_i(t) = u_i(e^t)$, then

$$(3.1) \qquad\qquad \tilde{u}_i'(t) \to -m_i \quad \text{as} \quad t \to \infty.$$

The equation for $\tilde{u}_i(t)$ is

$$(3.2) \qquad \tilde{u}_i''(t) + \sum_{j=1}^{n} a_{ij} e^{(2+\beta_j)t + \tilde{u}_j(t)} = 0, \quad t \in \mathbb{R}, \quad i \in I.$$

Let $z_i(t) = \sum_{j=1}^{n} a^{ij} \tilde{u}_j(t)$, then $z_i'(t) \to -\sigma_i$ as $t \to \infty$. (3.2) can be rewritten as

$$(3.3) \qquad z_i''(t) = -e^{(2+\beta_i)t + \sum_{j=1}^{n} a_{ij} z_j}, \quad i \in I.$$

Clearly $z_i''(t) < 0$ for all $i \in I$ and all $t \in \mathbb{R}$. Let $w_i(t) = z_i'(t)$, then by (3.3) and (3.1)

$$w_i(-\infty) = 0, \quad w_i(t) < 0 \; \forall t, \quad w_i(\infty) = -\sigma_i.$$

In addition we have $w_i'(-\infty) = w_i'(\infty) = 0$. Using the definition of w_i we differentiate (3.3) to obtain

$$(3.4) \qquad w_i''(t) = (2 + \beta_i) w_i'(t) + w_i'(t) \sum_{j \in I} a_{ij} w_j(t).$$

Taking the summation for $i \in I$ in (3.4) we can write (3.4) as

$$\sum_{i \in I} w_i''(t) - \sum_{i \in I} (2 + \beta_i) w_i'(t) = \sum_{i,j \in I} \frac{1}{2} a_{ij} (w_i(t) w_j(t))'.$$

Integrating t from $-\infty$ to ∞ we obtain $\Lambda_I(\sigma) = 0$. For $J \subset I$, summation for $i \in J$ in (3.4) leads to

$$\sum_{i \in J} w_i''(t) - \sum_{i \in J} (2 + \beta_i) w_i'(t) - \frac{1}{2} \sum_{i,j \in J} a_{ij} (w_i w_j)'(t) = \sum_{i \in J, j \in I \setminus J} a_{ij} w_i'(t) w_j(t).$$

Integrating the above for $t \in (-\infty, \infty)$ we obtain

$$\sum_{i \in J} (2 + \beta_i) \sigma_i - \frac{1}{2} \sum_{i,j \in J} a_{ij} \sigma_i \sigma_j = \sum_{i \in J, j \in I \setminus J} a_{ij} \int_{-\infty}^{\infty} w_i'(t) w_j(t) dt.$$

Since the irreducibility of A means that there exists $a_{ij} > 0$ for $i \in J, j \in I \setminus J$ we see the right hand side of the above is strictly positive because $w_i'(t) < 0$ for all $i \in I$ and $t \in \mathbb{R}$ and $w_i(t) < 0$ for all $i \in I$ and $t \in \mathbb{R}$. Thus we have obtained $\Lambda_J(\sigma) > 0$. Lemma 3.1 is established. $\square$

3.2. **The proof of the third statement of Theorem 1.1.**

Proposition 3.1. *Let u and v both be radial solutions to (1.1) such that*

$$\int_0^\infty r^{\beta_i+1} e^{u_i(r)} dr = \int_0^\infty r^{\beta_i+1} e^{v_i(r)} dr, \quad i \in I.$$

Then $u_i(r) = v_i(\delta r) + (2 + \beta_i) \log \delta$ for some $\delta > 0$ and all $i \in I$.

To prove Proposition 3.1 we first establish a uniqueness result for the linearized system:

Lemma 3.2. *Let $\phi = (\phi_1, ..., \phi_n)$ be a bounded solution of (2.1), then $\phi_i(r) = C(ru_i'(r) + 2 + \beta_i)$ for all $i \in I$.*

Proof of Lemma 3.2: Let

$$\phi^0 = (ru_1'(r) + 2 + \beta_1, ..., ru_n'(r) + 2 + \beta_n).$$

Then by computation ϕ^0 is a solution to the linearized system. Suppose there exists another bounded solution ϕ^1 which is not a multiple of ϕ^0. Without loss of generality we assume $\phi_1^1(0) = 0$, as by Lemma 2.5 one of $\phi_i^1(0)$ must be different from $2 + \beta_i$. To derive a contradiction we set

$$S = \{\alpha; \quad \exists \text{ a bounded solution } \phi = (\phi_1, ..., \phi_n) \text{ such that } \phi_1(0) = 2 + \beta_1,$$
$$\phi_i(0) = \alpha_i \le 3 + \beta_i;, \quad i = 2, ..., n, \quad \alpha = \min\{2 + \beta_1, \alpha_2, ..., \alpha_n\},$$
$$\int_0^r e^{u_i(s)} \phi_i(s) s^{1+\beta_i} ds > 0, \quad \forall r > 0, \quad i = 1, ..., n. \quad \}.$$

First we see that $2 + \min\{\beta_1, ..., \beta_n\} \in S$. Indeed the expression of ϕ^0 gives

$$\int_0^r s^{1+\beta_i} e^{u_i(s)} \phi_i^0(s) ds = r^{2+\beta_i} e^{u_i(r)} > 0.$$

Next we observe that S is a bounded set. Indeed, suppose $\alpha < 0$ is in S, let $\tilde{\phi}$ be the function corresponding to α, then $\exists j \in I$ such that $\tilde{\phi}_j(0) = \alpha$. This leads to $\int_0^r s^{1+\beta_j} e^{u_j(s)} \tilde{\phi}_j(s) ds < 0$ for r small, a contradiction to the definition of S. Let $\bar{\alpha}$ be the infimum of S and let $\alpha^k = (\alpha_1^k, ..., \alpha_n^k) \in S$ be a sequence in S that tends to $\bar{\alpha}$ from above. Suppose $\phi^k = (\phi_1^k, ..., \phi_n^k)$ is the solution corresponding to α^k, then we claim that ϕ^k converges to $\bar{\phi} = (\bar{\phi}_1, ..., \bar{\phi}_n)$, which is also a bounded solution with the strict monotonicity property described in S. Indeed, let $\psi^m = (\psi_1^m, ..., \psi_n^m)$ be the solution to the linearized system such that $\psi_i^m(0) = \delta_i^m$. Then by Lemma 2.5

$$\phi^k = \sum_{m=1}^n \alpha_m^k \psi^m.$$

Here we recall that by Lemma 2.2 $\psi_i^m(r) = O(\log r)$ for r large. Since $\bar{\alpha} \le \alpha_i^k \le 3 + \beta_i$ for $i \in I$ and all k. Along a subsequence ϕ^k tends to $\bar{\phi}$ over all compact subsets of $\mathbb{R}$. The monotonicity property of ϕ^k implies

$$\int_0^r e^{u_i(s)} \bar{\phi}_i(s) s^{1+\beta_i} ds \ge 0, \quad \forall i \in I, \quad \forall r > 0.$$

On the other hand, since ϕ^k are all bounded functions, for each ϕ_i^k we find $r_l \to \infty$ as $l \to \infty$ such that $r_l(\phi_i^k)'(r_l) \to 0$. From the equation for $r\phi_i^k$ we have

$$\int_0^\infty \sum_{j=1}^n a_{ij} r^{\beta_j+1} e^{u_j(r)} \phi_j^k(r) dr = 0, \quad \forall i \in I.$$

Since A is invertible

$$0 = \int_0^\infty e^{u_i(r)} \phi_i^k(r) r^{\beta_i+1} dr = \sum_{m=1}^n \alpha_m^k \int_0^\infty e^{u_i(r)} \psi_i^m(r) r^{\beta_i+1} dr.$$

Since $\psi_i^m(r) = O(\log r)$, $\int_0^\infty e^{u_i(r)} \psi_i^m(r) r^{\beta_i+1} dr$ is well defined, we let $\alpha^k \to (\bar\alpha_1, ..., \bar\alpha_n)$ to obtain

(3.5)
$$\int_0^\infty e^{u_i(s)} \bar\phi_i(s) s^{\beta_i+1} ds = 0, \quad \forall i \in I.$$

As a consequence of (3.5), $\bar\phi$ is bounded. Indeed, the equation for $\bar\phi$ is

$$(r\bar\phi_i'(r))' = -\sum_j a_{ij} r^{\beta_j+1} e^{u_j(r)} \bar\phi_j(r), \quad r > 0.$$

Using $\bar\phi_i(r) = O(\log r)$, $r^{\beta_i+2} e^{u_i(r)} = O(r^{-\delta})$ for some $\delta > 0$ (Lemma 2.1) and (3.5) we know

$$\int_0^r e^{u_i(s)} \bar\phi_i(s) s^{\beta_i+1} ds = 0 - \int_r^\infty e^{u_i(s)} \bar\phi_i(s) s^{\beta_i+1} ds = O(r^{-\delta/2})$$

for r large. Thus $\bar\phi_i'(r) = O(r^{-1-\delta})$ for all r large, which implies that $\bar\phi_i$ is bounded. Since each $\bar\phi_i$ is a non-increasing function, (3.5) implies that $\bar\phi_i$ decreases to a negative constant when $r \to \infty$. Indeed, by (3.5) either $\bar\phi_i \equiv 0$ or $\bar\phi_i$ decreases to a negative constant. The first possibility does not exist, because the fact $\bar\phi_1(0) = 2 + \beta_1 > 0$ implies that $\bar\phi_1$ decreases into a negative constant at infinity. Also $\int_0^r s^{1+\beta_1} e^{u_1(s)} \bar\phi_i(s) ds > 0$ for all r. Consequently for all i in the set $I_1 := \{i \in I; \ a_{i1} > 0\}$,

$$r\bar\phi_i'(r) \le -a_{i1} \int_0^r s^{1+\beta_1} e^{u_1(s)} \bar\phi_1(s) ds < 0, \quad \forall r > 0.$$

Therefore $\bar\phi_i$ strictly decreases to a negative constant for all $i \in I_1$. We can further define

$$I_2 := \{i \in I; \ \ a_{ij} > 0 \ \text{ for some } j \in I_1. \}.$$

By the same reason as above $\bar\phi_i$ decreases to a negative constant at infinity for all $i \in I_2$. By the irreducibility of A all the components of $\bar\phi$ decrease to negative constants at infinity.

Now we claim that $\bar\alpha - \epsilon \in S$ for $\epsilon > 0$ small. To see this, consider $\bar\phi + t\phi^1$ for $|t|$ sufficiently small. Recall that $\phi_1^1(0) = 0$, thus $\bar\phi_1(0) + t\phi_1^1(0) = 2 + \beta_1$.

Clearly $\bar{\phi} + t\phi^1$ solves (2.1). By choosing t positive or negative with $|t|$ small we can make

$$\min_{i \in I} \bar{\phi}_i(0) + t\phi_i^1(0) = \bar{\alpha} - \epsilon > 0.$$

Since $\bar{\phi} + t\phi^1$ is bounded we have

$$\int_0^\infty e^{u_i}(\bar{\phi}_i + t\phi_i^1)s^{\beta_i+1}ds = 0, \quad i = 1, ..., n.$$

Since $\bar{\phi}_i(r)$ tends to a negative constant as $r \to \infty$ and ϕ^1 is bounded, we know for r large and $|t|$ small

$$\int_r^\infty e^{u_i}(\bar{\phi}_i(s) + t\phi_i^1(s))s^{\beta_i+1}ds < 0.$$

Consequently

$$\int_0^r e^{u_i(s)}(\bar{\phi}_i(s) + t\phi_i^1(s))s^{\beta_i+1}ds > 0 \quad \forall r > 0.$$

Thus $\bar{\alpha} - \epsilon \in S$ for some $\epsilon > 0$ small, a contradiction to the definition of $\bar{\alpha}$. Lemma 3.2 is established. $\square$

Proof of Proposition 3.1: We shall consider

$$(3.6) \quad \begin{cases} u_i''(r) + \frac{1}{r}u_i'(r) + \sum_{j \in I} a_{ij}r^{\beta_j}e^{u_j(r)} = 0, \quad 0 < r < \infty, \\ \\ \int_0^\infty e^{u_i(r)}r^{\beta_i+1}dr < \infty, \quad \forall i \in I, \\ \\ u_i(0) = c_i, \quad i = 1, ..., n-1, \quad u_n(0) = 0. \end{cases}$$

Let

$$\Pi_2 := \{\sigma = (\sigma_1, .., \sigma_n); \quad \Lambda_I(\sigma) = 0, \quad \Lambda_J(\sigma) > 0, \quad \forall \emptyset \subsetneq J \subsetneq I. \quad \}.$$

$$\Pi_1 := \{\mathbf{C} = (c_1, .., c_{n-1}); \quad (3.6) \text{ has a solution.} \quad \}.$$

Note that by Lemma 2.4 $\Pi_1 = \mathbb{R}^{n-1}$ if $a_{ii} > 0$ for all i. We claim that the mapping from Π_1 to Π_2 is locally one to one. Indeed, let $\mathbf{M}$ be the following matrix:

$$\mathbf{M} = \begin{pmatrix} \partial_{c_1}\sigma_1 & ... & \partial_{c_{n-1}}\sigma_1 \\ ... & ... & ... \\ \partial_{c_1}\sigma_{n-1} & ... & \partial_{c_{n-1}}\sigma_{n-1} \end{pmatrix}$$

We claim that $\mathbf{M}$ is nonsingular. We prove this claim by contradiction. Suppose there exists a non-zero vector $\mathbf{D} = (d_1, ..., d_{n-1})^T$ such that $\mathbf{MD} = 0$. Then by setting $\gamma = d_1c_1 + ... + d_{n-1}c_{n-1}$ we have

$$(3.7) \qquad \partial_\gamma \sigma_1 = \partial_\gamma \sigma_2 = ... = \partial_\gamma \sigma_{n-1} = 0.$$

For Π_2, $\Lambda_I(\sigma) = 0$ reads

$$\sum_{i,j \in I} a_{ij}\sigma_i\sigma_j = 2\sum_{i \in I}(2 + \beta_i)\sigma_i.$$

By differentiating both sides with respect to γ we have

$$\sum_i (\sum_j a_{ij}\sigma_j - 2 - \beta_i)\partial_\gamma \sigma_i = 0.$$

Since $\Lambda_J(\sigma) > 0$ implies $\sum_j a_{ij}\sigma_j > 2 + \beta_i$, (3.7) implies $\partial_\gamma \sigma_n = 0$. Set $\phi_i = \partial_\gamma u_i$ $(i \in I)$, then $\phi = (\phi_1, ..., \phi_n)$ satisfies (2.1) and

$$\phi_i(0) = d_i, \quad i = 1, ..., n - 1, \quad \phi_n(0) = 0.$$

From $\partial_\gamma \sigma_i = 0$ $(i \in I)$ we have

(3.8) $$\int_0^\infty e^{u_i}\phi_i(s)s^{1+\beta_i}ds = 0, \quad i \in I.$$

As a consequence of (3.8), ϕ is bounded. Indeed, integrating (2.1) from 0 to r

$$r\phi_i'(r) = -\int_0^r \sum_j a_{ij}s^{1+\beta_j}e^{u_j(s)}\phi_j(s)ds$$

$$= \int_r^\infty a_{ij}s^{1+\beta_j}e^{u_j(s)}\phi_j(s)ds = O(r^{-\delta})$$

for some $\delta > 0$. Therefore $\phi'(r) = O(r^{-1-\delta})$, which proves that ϕ_i is bounded. By Lemma 3.2 $\phi_i = c(ru_i' + 2 + \beta)$, then we see immediately that $c = 0$ because $\phi_n(0) = 0$, this is not possible because not all d_i's are zero. Therefore we have proved that $\mathbf{M}$ is nonsingular for all $\mathbf{C} = (c_1, ..., c_{n-1}) \in \Pi_1$.

We further assert that there is one-to-one correspondence between Π_1 and Π_2. This is proved in two steps as follows.

Case 1: $a_{ii} > 0$, $i \in I$.

In this case, by Lemma 2.4 $\Pi_1 = \mathbb{R}^{n-1}$. The mapping from Π_1 to Π_2 is proper and locally one to one. Here we claim that Π_2 is simply connected. Assuming this, since $\mathbb{R}^{n-1}$ and Π_2 are simply connected, there is one to one correspondence between them. Let $u = (u_1, ..., u_n)$ and $v = (v_1, .., v_n)$ be two radial solutions such that $u_n(0) = v_n(0) = 0$, $\int_{\mathbb{R}^2} |x|^{\beta_i}e^{u_i} = \int_{\mathbb{R}^2} |x|^{\beta_i}e^{v_i}$ $(i \in I)$. Then $u_i(0) = v_i(0)$ for $i = 1, ..., n - 1$. By Lemma 2.3 $u_i \equiv v_i$ for all $i \in I$. Now we prove that Π_2 is simply connected. Indeed, using $m_i = \sum_j a_{ij}\sigma_j$, $\Lambda_I(\sigma) = 0$ can be written as

(3.9) $$\sum_{i,j\in I} a^{ij}(2 + \beta_i)(2 + \beta_j) = \sum_{i,j\in I} a^{ij}(m_i - 2 - \beta_i)(m_j - 2 - \beta_j).$$

Therefore Π_2 is part of a quadratic surface, the boundary of which is restricted by $\Lambda_{J_i}(\sigma) = 0$ where J_i is I with the index i removed. $\Lambda_{J_i}(\sigma) > 0$ reads

$$m_i - 2 - \beta_i > \frac{a_{ii}}{2}\sigma_i.$$

In another word in the coordinate system represented by m_i, we use n coordinate planes to bound the quadratic hypersurface described in (3.9). Other restrictions $\Lambda_J > 0$, when J is obtained from I with at least two indices

removed, do not affect the topological information of $\Lambda_I(\sigma) = 0$. Thus Π_2 is a part of the quadratic hyper-surface in the first quadrant and is therefore simply connected. Proposition 3.1 is proved in this case.

Case 2: There exists i_0 such that $a_{i_0,i_0} = 0$. We prove this case by a contradiction. Suppose $c^k = (c_1^k, ..., c_{n-1}^k)$ $(k = 1, 2)$ are two distinct points on Π_1 that correspond to the same energy: let u^1, u^2 be two solutions corresponding to c^1 and c^2 respectively such that

$$\int_0^\infty e^{u_i^1(r)} r^{1+\beta_i} dr = \int_0^\infty e^{u_i^2(r)} r^{1+\beta_i} dr = \sigma_i, \quad i \in I.$$

Since the matrix $\left(\dfrac{\partial \sigma}{\partial c}\right) \in \mathbb{M}_{(n-1)\times(n-1)}$ is nonsingular at c^1 and c^2, there is a one-to-one mapping between a neighborhood of c^k to a neighborhood of σ in Π_2. Since $c^1 \neq c^2$, we choose the neighborhoods around them to be disjoint.

Now consider a perturbation system

$$(3.10) \quad \begin{cases} u_i''(r) + \frac{1}{r} u_i'(r) + \sum_{j\in I}(a_{ij} + \epsilon\delta_{ij}) r^{\beta_j} e^{u_j} = 0, \quad r > 0, \quad i \in I, \\ \int_0^\infty r^{\beta_i+1} e^{u_i} dr < \infty, \quad i \in I, \\ u_1(0) = c_1, ...u_{n-1}(0) = c_{n-1}, \quad u_n(0) = 0. \end{cases}$$

Let $u^{k,\epsilon}$ be the solution to (3.10) that corresponds to the initial condition $c^k = (c_1^k, ..., c_{n-1}^k, 0)$ $(k = 1, 2)$. Let $\sigma^{k,\epsilon} = (\sigma_1^{k,\epsilon}, .., \sigma_n^{k,\epsilon})$ be defined as $\sigma_i^{k,\epsilon} = \int_0^\infty r^{\beta_i+1} e^{u_i^{k,\epsilon}(r)} dr$ $(i = 1, .., n)$. We claim that

$$(3.11) \qquad\qquad \sigma^{k,\epsilon} = (\sigma_1, .., \sigma_n) + \circ(1), \quad k = 1, 2.$$

and

$$(3.12) \qquad \frac{\partial \sigma_i^{k,\epsilon}}{\partial c_j} = \frac{\partial \sigma_i}{\partial c_j} + \circ(1), \quad i = 1, .., n, \quad j = 1, .., n-1, \quad k = 1, 2.$$

Assuming (3.11) and (3.12) for the moment. Now the matrix

$$\begin{pmatrix} \partial_{c_1}\sigma_1^{k,\epsilon} & \cdots & \partial_{c_{n-1}}\sigma_1^{k,\epsilon} \\ \vdots & \vdots & \vdots \\ \partial_{c_1}\sigma_{n-1}^{k,\epsilon} & \cdots & \partial_{c_{n-1}}\sigma_{n-1}^{k,\epsilon} \end{pmatrix}$$

is non-singular at c^k $(k = 1, 2)$ for ϵ small. On the other hand, $\sigma^{1,\epsilon}$ and $\sigma^{2,\epsilon}$ both satisfy

$$(3.13) \quad \begin{cases} \Lambda_I^\epsilon(\sigma^{k,\epsilon}) := \sum_{i\in I} 2(2+\beta_i)\sigma_i^{k,\epsilon} - \sum_{i,j\in I}(a_{ij} + \epsilon\delta_{ij})\sigma_i^{k,\epsilon}\sigma_j^{k,\epsilon} = 0 \\ \Lambda_J^\epsilon > 0, \quad 0 \subsetneqq J \subsetneqq I. \end{cases}$$

We use Π^ϵ to represent the hyper-surface described as above. For $\sigma^{2,\epsilon} = (\sigma_1^{2,\epsilon}, .., \sigma_n^{2,\epsilon}) \in \Pi^\epsilon$, we can find $c^{1,\epsilon} = (c_1^{1,\epsilon}, .., c_{n-1}^{1,\epsilon})$ such that

$$c_j^{1,\epsilon} = c_j^1 + \circ(1), \quad j = 1, 2, .., n-1$$

and a solution $\bar{u}^{1,\epsilon}$ of (3.10) with the initial condition $(c_1^{1,\epsilon}, .., c_{n-1}^{1,\epsilon}, 0)$ such that

$$\int_0^\infty r^{\beta_j+1} e^{\bar{u}_j^{1,\epsilon}} dr = \sigma_j^{2,\epsilon}, \quad j = 1, 2, .., n-1.$$

After using $\Lambda_I^\epsilon(\sigma^{2,\epsilon}) = 0$ in (3.13) we have

$$\int_0^\infty r^{\beta_n+1} e^{\bar{u}_n^{1,\epsilon}} dr = \sigma_n^{2,\epsilon}.$$

Then the difference between c^1 and c^2 implies $c^{1,\epsilon} \neq c^2$ for ϵ small. A contradiction to the uniqueness property satisfied by the system (3.10).

To finish the proof we now verify (3.11) and (3.12). Here we require $\epsilon \in (0, \delta_0)$ where δ_0 is so small that the matrix $(a_{ij} + \epsilon\delta_{ij})_{n\times n}$ is non-singular for all $\epsilon \in (0, \delta_0)$.

For u^k, there exists R_0 large such that for $r > R_0$ and some $\delta > 0$,

$$(u_i^k)'(r)r \leq -2 - \beta_i - 2\delta, \quad i = 1, ..., n, \quad k = 1, 2.$$

For δ_0 small we have $u_i^{k,\epsilon}$ converges uniformly to u_i^k over $0 \leq r \leq R_0$. For $r = R_0$ we have

$$(u_j^{k,\epsilon}(r))'r \leq -(2 + \beta_j + \delta) \quad \text{at} \quad r = R_0, \quad 0 \leq \epsilon \leq \delta_0.$$

Then by the super-harmonicity of $u_j^{k,\epsilon}$ it is easy to show

$$(u_j^{k,\epsilon}(r))'r \leq -(2 + \beta_j + \delta) \quad \text{for} \quad r \geq R_0.$$

Thus, $\exists C > 0$ and $R_1 \geq R_0$ such that

$$(3.14) \qquad r^{\beta_j} e^{u_j^{k,\epsilon}(r)} \leq Cr^{-(2+\delta)} \quad \text{for} \quad r \geq R_1$$

Hence for $k = 1, 2$,

$$\sigma_j^\epsilon = \int_0^\infty e^{u_j^{k,\epsilon}(r)} r^{\beta_j+1} dr = \int_0^\infty e^{u_j^k(r)} r^{\beta_j+1} dr + o(1) = \sigma_j + \circ(1), \quad j = 1, .., n.$$

(3.11) is verified. To show (3.12)

$$(3.15) \qquad \frac{\partial \sigma_i^\epsilon}{\partial c_j} = \int_0^\infty r^{\beta_i+1} e^{u_i^{k,\epsilon}(r)} \frac{\partial u_i^{k,\epsilon}}{\partial c_j}(r) dr, \quad i = 1, .., n, \quad k = 1, 2.$$

$\frac{\partial u^{k,\epsilon}}{\partial c}$ satisfies the following linearized equation:

$$-\Delta\left(\frac{\partial u_i^{k,\epsilon}}{\partial c_l}\right) = \sum_{j=1}^2 (a_{ij} + \epsilon\delta_{ij}) r^{\beta_j} e^{u_j^{k,\epsilon}} \frac{\partial u_j^{k,\epsilon}}{\partial c_l}, \quad i = 1, .., n, \quad l = 1, ..., n-1.$$

By Lemma 2.2

$$(3.16) \qquad |\frac{\partial u_i^{k,\epsilon}}{\partial c_l}(r)| \leq C \ln r, \quad r \geq 2, \quad i = 1, .., n, \quad l = 1, .., n-1$$

where the constant C is independent of $\epsilon \in (0, \delta_0)$. Moreover, for any fixed $R > 0$, $\frac{\partial u_i^{1,\epsilon}}{\partial c_l}(r)$ converges uniformly to $\frac{\partial u_i^1}{\partial c_l}(r)$ over $0 < r < R$ with respect to ϵ. Using the decay estimates (3.14) and (3.16) in (3.15) we obtain (3.12) by elementary analysis. Proposition 3.1 is proved in all cases. $\square$

3.3. The proof of the second statement of Theorem 1.1.
Our proof is based on the uniqueness result and is completely different from the method employed in [22]. We divide the proof into two cases according to the diagonal entries of A.

Case one: $a_{ii} > 0$ for all $i \in I$.

In this case, by Lemma 2.4, for any $c_1, ..., c_{n-1} \in \mathbb{R}$, there exists a unique finite energy solution $u = (u_1, ..u_n)$ such that $u_i(0) = c_i$ for $i = 1, .., n-1$ and $u_n(0) = 0$. By Proposition 3.1 there is a bijection between the initial condition $(c_1, .., c_{n-1}, 0)$ and Π_2 (see the notation in the proof of Proposition 3.1). Thus Theorem 1.1 is proved in this case.

Case two: There exists $i_0 \in I$ such that $a_{i_0, i_0} = 0$.

Let $\sigma \in \Pi_2$, then for $\epsilon > 0$ we consider

$$\begin{cases} -\Delta u_i^\epsilon = \sum_{j \in I}(a_{ij} + \epsilon \delta_{ij})|x|^{\beta_j} e^{u_j^\epsilon(x)}, \quad \mathbb{R}^2, \\[2mm] u_i^\epsilon(0) = c_i^\epsilon, \quad i = 1, ..., n-1, \quad u_n^\epsilon(0) = 0, \\[2mm] \int_0^\infty r^{\beta_i+1} e^{u_i^\epsilon(r)} dr = \sigma_i^\epsilon, \quad i \in I \end{cases}$$

where $\sigma^\epsilon = (\sigma_1^\epsilon, ..., \sigma_n^\epsilon)$ is a point on the hyper-surface

$$\Pi_2^\epsilon := \{\sigma = (\sigma_1, ..., \sigma_n); \ \sigma_i > 0, \ \forall i \in I, \ \Lambda_I^\epsilon(\sigma) = 0, \ \Lambda_J^\epsilon(\sigma) > 0, \ \forall \emptyset \subsetneq J \subsetneq I\}$$

such that $\sigma^\epsilon \to \sigma$ as $\epsilon \to 0$. Here we recall that $\Lambda_I^\epsilon(\sigma)$ is defined as

$$\Lambda_I^\epsilon(\sigma) := 2 \sum_{i \in I}(2 + \beta_i)\sigma_i - \sum_{i,j \in I}(a_{ij} + \epsilon \delta_{ij})\sigma_i \sigma_j.$$

The vector $(c_1^\epsilon, ..., c_{n-1}^\epsilon, 0)$ is the initial condition corresponding to σ^ϵ. Now we claim that

$$(3.17) \qquad \max c_i^\epsilon \leq C \quad i = 1, ..., n-1$$

for some $C > 0$ independent of ϵ. Indeed, if this is not the case, without loss of generality we assume c_1^ϵ is the largest among c_i^ϵ and tends to infinity. Re-scale u^ϵ according to c_1^ϵ to make the maximum of all components at 0 equal to 0. The re-scaled system has to converge in $C_{loc}^2(\mathbb{R}^2)$ norm to a partial system. Indeed, the first component converges because all the components are bounded. The $n - th$ component tends to $-\infty$ because

the initial condition is 0 before the scaling and all components are non-increasing. Therefore for the limit function $v = (v_1, ..., v_n)$ without loss of generality we assume $v_{m+1} = ... v_n = 0$ for some $1 < m < n$. For $i = 1, ..., m$ we easily observe that

$$(3.18) \qquad \bar{\sigma}_i := \int_0^\infty r^{\beta_i+1} e^{v_i(r)} dr \leq \sigma_i, \quad i = 1, ..., m.$$

The reason is for each fixed $R > 0$ we have

$$\int_0^R r^{\beta_i+1} e^{v_i(r)} dr \leq \sigma_i^\epsilon + \circ(1), \quad i = 1, ..., m.$$

Clearly $(v_1, ..., v_m)$ satisfies

$$\begin{cases} \Delta v_i + \sum_{j=1}^m a_{ij} r^{\beta_j} e^{v_j} = 0, & i = 1, ..., m, \\ \\ \int_0^\infty r^{\beta_i+1} e^{v_i(r)} dr \leq \sigma_i, & i = 1, .., m. \end{cases}$$

By Lemma 2.1

$$(3.19) \qquad \sum_{j=1}^m a_{ij} \bar{\sigma}_j > 2 + \beta_i, \quad i = 1, ...m.$$

We claim that $\bar{\sigma} = (\bar{\sigma}_1, ..., \sigma_n)$ with $\bar{\sigma}_{m+1} = ... = \bar{\sigma}_n = 0$ satisfies $\Lambda_I(\bar{\sigma}) = 0$. Indeed, let $v_{m+1} = ... = v_n \equiv 0$ and $H_i = 1$ if $i = 1, ..., m$ and $H_i = 0$ for $i = m+1, ..., n$. Then the system for v can be written as

$$\Delta v_i + \sum_{j=1}^n a_{ij} r^{\beta_j} H_j e^{v_j} = 0, \quad i = 1, ..., n.$$

Apply the standard method to obtain the Pohozaev identity to the system above we have $\Lambda_I(\bar{\sigma}) = 0$. Let $J = \{1, ...m\}$ we have $\Lambda_J(\bar{\sigma}) = 0$. Let $z_i = \sigma_i - \bar{\sigma}_i$. From the definition of $\bar{\sigma}_i$ we know that $z_i \geq 0$ for $i = 1, ..., m$ Since $1 < m < n$ we have $\Lambda_J(\sigma) > 0$, $\Lambda_J(\sigma) - \Lambda_J(\bar{\sigma}) > 0$ gives

$$(3.20) \qquad \sum_{i \in J} \left((\sum_{j \in J} a_{ij}\sigma_j - (2+\beta_i))z_i + (\sum_{j \in J} a_{ij}\bar{\sigma}_j - (2+\beta_i))z_i \right) < 0.$$

Since $\sum_{j \in J} a_{ij}\bar{\sigma}_j > 2 + \beta_i$ for all $i \in J$, we also have $\sum_{j \in J} a_{ij}\sigma_j > 2 + \beta_i$ for all $i \in J$ because $\sigma_i \geq \bar{\sigma}_i$. Clearly (3.20) is impossible. (3.17) is proved. Similarly there is a lower bound for $c_1^\epsilon, ..., c_{n-1}^\epsilon$. As $\epsilon \to 0$, the u^ϵ converges to u that corresponds to σ. Theorem 1.1 is proved in both cases. $\square$

4. Proof of Theorem 1.2

The proof of Proposition 4.1 in [22] can be readily applied to prove Theorem 1.2. We include it for the convenience of readers.

For $\lambda > 0$, let $u_i^\lambda(x_1, x_2) = u_i(2\lambda - x_1, x_2)$. Set $\Sigma_\lambda = \{x \in \mathbb{R}^2; \quad x_1 > \lambda.\}$ and T_λ be the boundary of Σ_λ. The equation for $u^\lambda = (u_1^\lambda, ..., u_n^\lambda)$ is

$$(4.1) \qquad \Delta u_i^\lambda + \sum_{j \in I} a_{ij} |x^\lambda| e^{u_j^\lambda} = 0, \quad i \in I$$

where $x^\lambda = (2\lambda - x_1, x_2)$. Set $w_i^\lambda = u_i^\lambda - u_i$ to be defined in Σ_λ for $\lambda > 0$. For w_i^λ we have

$$\Delta w_i^\lambda + \sum_j a_{ij} |x|^{\beta_i} e^{\xi_j^\lambda} w_j^\lambda = -\sum_j a_{ij} (|x^\lambda|^{\beta_j} - |x|^{\beta_j}) e^{u_j^\lambda}$$

where

$$e^{\xi_i^\lambda} = \frac{e^{w_i^\lambda} - e^{w_i}}{w_i^\lambda - w_i} = \int_0^1 e^{w_i + t(w_i^\lambda - w_i)} dt.$$

Since $\beta_i \le 0$ for all $i \in I$,

$$(4.2) \qquad \Delta w_i^\lambda + \sum_j a_{ij} |x|^{\beta_i} e^{\xi_j^\lambda} w_j^\lambda \le 0.$$

Let $f = \log \log(|x| + 3)$, then

$$\Delta f(x) = \frac{3}{r(r+3)^2 \log(r+3)} - \frac{1}{(r+3)^2 \log^2(r+3)}.$$

Therefore for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$(4.3) \qquad \frac{\Delta f}{f} \le -\frac{1}{r^{2+\epsilon}}, \quad r > C(\epsilon).$$

Let $z_i^\lambda = w_i^\lambda / f$, then the following lemma holds.

Lemma 4.1. *There exists $R > 0$ independent of λ such that for $\lambda > 0$, if x_0 is a point where a negative minimum of $\min\{z_1^\lambda, .. z_n^\lambda\}$ is attained, then $x_0 \in B_R$.*

Proof of Lemma 4.1: From (4.2) we obtain

$$(4.4) \qquad \Delta z_i^\lambda + 2\nabla z_i^\lambda \frac{\nabla f}{f} + z_i^\lambda \frac{\Delta f}{f} + \sum_j a_{ij} e^{\xi_j^\lambda} z_j^\lambda \le 0.$$

Suppose $z_i^\lambda(x_0) = \min_j z_j^\lambda(x_0) < 0$ and x_0 is where the negative minimum for z_i^λ is attained. Here we note that the global minimum of z_i^λ should be attained. Indeed, by Lemma 2.1,

$$u_i(x) = -m_i \log |x| + c_i + O(|x|^{-\delta})$$

when $|x|$ is large. Thus, for $\lambda > 0$, since $|x^\lambda| < |x|$,

$$w_i^\lambda(x) = u_i^\lambda(x) - u_i(x) \ge O(|x|^{-\delta}), \quad |x| >> 1.$$

Thus $\lim_{|x|\to\infty} z_i^\lambda(x) \geq 0$. Let $J = \{j \in I; \quad z_j(x_0) \leq 0\}$. Here we observe that the image of the origin is not in J, because of the decay rate of u_i. We rewrite (4.4) as

$$(4.5) \qquad \Delta z_i^\lambda + 2\nabla z_i^\lambda \frac{\nabla f}{f} + z_i^\lambda \frac{\Delta f}{f} + \sum_{j \in J} a_{ij} e^{\xi_j^\lambda} z_j^\lambda \leq 0.$$

in a small neighborhood of x_0. Then at x_0,

$$\Delta z_i^\lambda(x_0) \geq 0, \quad 2\nabla z_i^\lambda(x_0) \frac{\nabla f(x_0)}{f(x_0)} = 0.$$

For $j \in J$, since $w_j^\lambda(x_0) \leq 0$, we have $u_j^\lambda(x_0) \leq u_j(x_0)$, so if $|x_0|$ is large, by Lemma 2.1, $e^{\xi_j^\lambda(x)} \sim |x|^{-2-\delta}$ for x close to x_0 and some $\delta > 0$. Thus

$$\sum_{j \in J} a_{ij} e^{\xi_j^\lambda(x_0)} z_j^\lambda(x_0) \leq z_i(x_0) \sum_j a_{ij} e^{\xi_j^\lambda(x_0)} \leq C z_i^\lambda(x_0) |x_0|^{-2-\delta}.$$

On the other hand if $|x_0|$ is large

$$z_i^\lambda(x_0) \frac{\Delta f(x_0)}{f(x_0)} > |z_i^\lambda(x_0)| |x_0|^{-2-\epsilon}.$$

Therefore by choosing $\epsilon < \delta/2$ we see that (4.5) can not hold if $|x_0|$ is large. Lemma 4.1 is established. $\square$

By Lemma 4.1 and Lemma 2.1, $\min\{w_1^\lambda, ..., w_n^\lambda\} > 0$ in Σ_λ for λ sufficiently large. Thus set

$$\bar{\lambda} := \inf\{\lambda > 0; \quad \min\{w_1^\lambda, ..., w_n^\lambda\} > 0 \quad \text{in } \Sigma_\lambda \ \}.$$

Lemma 4.2. $\bar{\lambda} = 0$.

Proof of Lemma 4.2: If $\bar{\lambda} > 0$, we first prove that $w_i^{\bar{\lambda}} > 0$ in $\Sigma_{\bar{\lambda}}$ for all $i \in I$. Indeed, let $I_0 = \{i \in I; \quad w_i^{\bar{\lambda}} \equiv 0\}$. If I_0 is not empty, the irreducibility of A implies all $w_i^{\bar{\lambda}} \equiv 0$ in $\Sigma_{\bar{\lambda}}$. However, not all β_i are 0, so for some $i \in I$, we have

$$\Delta w_i^{\bar{\lambda}} + \sum_{j \in I} a_{ij} |x|^{\beta_j} e^{\xi_j^{\bar{\lambda}}} w_j^{\bar{\lambda}} = -\sum_{j \in I} a_{ij} (|x^{\bar{\lambda}}|^{\beta_j} - |x|^{\beta_j}) e^{u_j^{\bar{\lambda}}} < 0.$$

A contradiction.

Next we derive a contradiction to the definition of $\bar{\lambda}$. Let λ_k tend to $\bar{\lambda}$ from the left. Thus $\lambda_k > 0$ for all large k. We can assume that $\min_{i \in I} w_i^{\lambda_k} < 0$ in Σ_{λ_k} because otherwise, the strong maximum principle implies $w_i^{\lambda_k} > 0$ in Σ_{λ_k}, a contradiction to the definition of $\bar{\lambda}$. Therefore, let x_k be where the minimum of $\min_{i \in I} w_i^\lambda$ be attained and there is $i_k \in I$ such that $w_{i_k}^{\lambda_k}(x_k) = \min_{i \in I, x \in \Sigma_{\lambda_k}} w_i^{\lambda_k} < 0$. By Lemma 4.1, $x_k \in B_R$ for some $R > 0$ and all k. Along a subsequence $\{x_k\}$ converges to $\bar{x} \in \Sigma_{\bar{\lambda}}$ such that for some $i_0 \in I$, $w_{i_0}^{\bar{\lambda}}(x_0) = 0$. Since we have proved that $w_i^{\bar{\lambda}} > 0$ for all $i \in I$ in $\Sigma_{\bar{\lambda}}$, $x_0 \in T_{\bar{\lambda}}$.

However, $\nabla w_{i_k}(x_k) = 0$ leads to $\nabla w_{i_0}^{\bar\lambda}(x_0) = 0$, a contradiction to the Hopf Lemma. Lemma 4.2 is established. $\square$

Thus we have proved $\bar\lambda = 0$, which leads to

$$u_i(-x_1, x_2) \geq u_i(x_1, x_2), \quad \forall x_1 \geq 0, \quad i \in I.$$

Moving the plane from all possible directions we obtain the symmetry of u_i. Theorem 1.2 is established. $\square$

5. Uniqueness theorem on the linearized system

In this section we prove Theorem 1.3. The following lemma describes the projection of u on $\sin k\theta$ and $\cos k\theta$.

Lemma 5.1. *Let $\phi_{i,k}(r)$ satisfy*

$$(5.1) \qquad \phi_{i,k}'' + \frac{1}{r}\phi_{i,k}' + \sum_{j \in I} a_{ij} r^{\beta_j} e^{u_j(r)} \phi_{j,k} - \frac{k^2}{r^2}\phi_{i,k} = 0, \quad 0 < r < \infty$$

and

$$(5.2) \qquad |\phi_{i,k}(r)| \leq Cr^k(1+r)^{-2k}, \quad \forall r > 0, \quad k \geq 1.$$

If there exists $f = (f_1, .., f_n)$ such that

$$(5.3) \qquad f_i'' + \frac{1}{r}f_i' - \frac{k^2}{r^2}f_i + \sum_{j=1}^n a_{ij} r^{\beta_j} e^{u_j} f_j < 0, \quad r > 0$$

and

$$(5.4) \qquad f_i(r) > 0, \quad \forall r > 0, \quad \lim_{r \to 0} f_i(r)/r^k = \infty, \quad \lim_{r \to \infty} f_i(r) r^k = \infty.$$

Then $\phi_{ik} \equiv 0$.

Proof of Lemma 5.1: We only need to show $\phi_{ik} \leq 0$. Suppose this is not the case. Then because of the assumptions on the decay rates, without loss of generality we assume

$$(5.5) \qquad w_1(r_0) = \frac{\phi_{1,k}(r_0)}{f_1(r_0)} = \max_{i,r} \frac{\phi_{i,k}(r)}{f_i(r)} > 0.$$

Note that the maximum can be attained because of the decay assumptions on $\phi_{i,k}$ and (5.4). The equation for w_1, after simple derivation, is

$$w_1'' + \left(\frac{2f_1'}{f_1} + \frac{1}{r}\right)w_1' + w_1\left(\frac{f_1''}{f_1} + \frac{1}{r}\frac{f_1'}{f_1} - \frac{k^2}{r^2} + a_{11}r^{\beta_1}e^{u_1}\right)$$

$$+ \sum_{j=2}^n a_{1j} r^{\beta_k} e^{u_j} \frac{\phi_{k,j}}{f_1} = 0.$$

Near r_0, $w_1(r) > 0$. Thus in the neighborhood of r_0, using (5.3) we have

$$w_1'' + \left(\frac{2f_1'}{f_1} + \frac{1}{r}\right)w_1' > \sum_{j=2}^n a_{1j} r^{\beta_j} e^{u_j} \frac{f_j w_1 - \phi_{k,j}}{f_1}.$$

The left hand side of the above is non-positive when evaluated at r_0, while the right hand side is non-negative. A contradiction. Lemma 5.1 is established. $\square$

Proof of Theorem 1.3:

Let $f_i = -u_i'(r)$. Direct computation shows that

$$f_i'' + \frac{1}{r}f_i' - \frac{1}{r^2}f_i + \sum_j a_{ij}r^{\beta_j}e^{u_j}f_j = \sum_j a_{ij}\beta_j r^{\beta_j-1}e^{u_j}.$$

Since all $\beta_i < 0$. $f = (f_1, ..., f_n)$ satisfies (5.3) and (5.4) for all $k \geq 2$. Let $\phi^k = (\phi_1^k, ..., \phi_n^k)$ be the radial part of the projection onto, say, $\sin k\theta$. Then ϕ^k satisfies (5.1). Since ϕ^k is bounded, it is easy to apply standard ODE theorem to obtain that (5.2) also holds. Thus all the projections on $\sin k\theta$ and $\cos k\theta$ are all zero for $\beta \leq 0$.

Finally we prove that for the projection on $\sin\theta$ or $\cos\theta$ is also zero. Let $\phi^1 = (\phi_{1,1}, .., \phi_{1,n})$ be the projection of ϕ on $\sin\theta$. Then we have

$$\phi_{1,i}'' + \frac{1}{r}\phi_{1,i}' - \frac{1}{r^2}\phi_{1,i} + \sum_j a_{ij}r^{\beta_j}e^{u_j}\phi_{1,j} = 0.$$

Since ϕ^1 is bounded, the standard ODE theory implies that $\phi_{1,i}$ behaves like $O(1/r)$ at infinity and like $O(r)$ near 0. We shall use $f = (-u_1', ..., -u_n')$ as the function to majorize ϕ^1. To apply the same argument as in the proof of Lemma 5.1, The problem is the maximum may tend to 0 or infinity. We first prove that this can not happen at 0:

(5.6) ϕ_i/f_i is strictly increasing near 0 if ϕ_i is positive near 0.

Clearly once (5.6) is proved, $\phi^1 \equiv 0$, thus Theorem 1.3 would be established.

Now we prove (5.6). Let $z_i = \phi_{1,i}/r$ and $F_i = f_i/r$. Direct computation yields

$$z_i'' + \frac{3}{r}z_i' + \sum_j a_{ij}r^{\beta_j}e^{u_j}z_j = 0, \quad r > 0.$$

and

$$F_i'' + \frac{3}{r}F_i' + \sum_j a_{ij}r^{\beta_j}e^{u_j}F_j = \sum_j a_{ij}\beta_j r^{\beta_j-2}e^{u_j}.$$

Since $\phi_{1,i}$ is positive near 0, $z_i(0) > 0$ (if $z_i(0) = 0$, there is no need to consider this case, as the maximum can not tend to 0). Easy to see that near 0,

$$z_i(r) = z_i(0) + \sum_j O(r^{\beta_j+2})$$

and

$$F_i(r) = F_i(0) + \sum_j \frac{a_{ij}}{\beta_j+2}r^{\beta_j}e^{u_j(0)} + \sum_j O(r^{\beta_j+1}).$$

Proving $\phi_{1,i}/f_i$ to be increasing near 0 is the same as proving that z_i/F_i is increasing near 0. Since $\beta_i < 0$, one immediately sees that z_i/F_i is increasing near 0.

Next we prove that z_i/F_i is decreasing if z_i is positive at infinity. Assume $z_i(r) = q_i/r^2 + O(r^3)$ at infinity. We have known that, for some $\delta_i > 0$,

$$u_i(r) = -m_i \log r + c_i + O(r^{-\delta_i}) \quad r > 1.$$

Thus

$$e^{u_i(r)} = e^{c_i} r^{-m_i} + O(r^{-m_i - \delta_i}), \quad r >> 1.$$

We obtain, by integration on the equation for z_i, that

$$z_i'(r) = -\frac{2q_i}{r^3} + \sum_j a_{ij} \frac{e^{c_j} q_j}{m_j - \beta_j - 2} r^{\beta_j - m_j - 1} + O(r^{\beta_j - m_j - 1 - \delta_j}).$$

$$z_i(r) = \frac{q_i}{r^2} - \sum_j a_{ij} \frac{e^{c_j} q_j}{(m_j - \beta_j - 2)(m_j - \beta_j)} r^{\beta_j - m_j} + \sum_j O(r^{\beta_j - m_j - \delta_j}).$$

Correspondingly to compute F_i, we use the equation for u_i to obtain

$$(ru_i'(r))' = -\sum_j a_{ij} r^{\beta_j + 1} e^{u_j} = -\sum_j a_{ij} r^{1 + \beta_j - m_j} e^{c_j} + O(r^{1 + \beta_j - m_j - \delta_j}).$$

Using $ru_i'(r) \to -m_i$ at infinity, we have

$$ru_i'(r) = -m_i + \sum_j a_{ij} \frac{e^{c_j}}{m_j - \beta_j - 2} r^{2 + \beta_j - m_j} + O(r^{2 + \beta_j - m_j - \delta_j}).$$

Consequently

$$F_i = -\frac{u_i'(r)}{r} = \frac{m_i}{r^2} - \sum_j a_{ij} \frac{e^{c_j}}{m_j - \beta_j - 2} r^{\beta_j - m_j} + O(r^{\beta_j - m_j - \delta_j}).$$

$$F_i' = -\frac{2m_i}{r^3} + \sum_j a_{ij} e^{c_j} \frac{m_j - \beta_j}{m_j - \beta_j - 2} r^{\beta_j - m_j - 1} + O(r^{\beta_j - m_j - 1 - \delta_j}).$$

Our goal is

$$z_i' F_i - z_i F_i' < 0 \text{ near infinity}$$

when z_i is positive near infinity. Using the expressions above it is enough to show if the following is negative:

(5.7) $$\sum_j a_{ij} e^{c_j} (-q_i + \frac{m_i}{m_j - \beta_j} q_j) r^{\beta_j - m_j - 2}.$$

By $q_i/m_i = \max_{j \in I} q_j/m_j$, $q_i > 0$ and $\beta_i < 0$ for all $i \in I$, we have (5.7). Therefore z_i/F_i is decreasing near infinity. Theorem 1.3 is proved. $\square$

6. Appendix

In this appendix, we prove the ODE lemmas stated in section two.
Proof of Lemma 2.1: The proof is standard (for example, see [20]). We include it for the convenience of the reader. Let

$$(6.1) \quad w_i(x) = \int_{\mathbb{R}^2} (-\frac{1}{2\pi} \log|x-y| + \frac{1}{2\pi} \log(1+|y|)) \sum_j a_{ij}|y|^{\beta_j} e^{u_j(y)} dy.$$

Clearly w_i is well defined and satisfies

$$-\Delta w_i(x) = \sum_j a_{ij}|x|^{\beta_j} e^{u_j(x)}, \quad \mathbb{R}^2$$

and

$$\Delta(u_i - w_i) = 0, \quad \mathbb{R}^2.$$

By Lemma 4.1 in [40] $u_i \le C$ on $\mathbb{R}^2$. Next we claim that

$$\lim_{|x| \to \infty} \frac{w_i(x)}{\log|x|} = -m_i.$$

To see the above, it is easy to obtain for $\epsilon > 0$, there exist $R(\epsilon) >> 1$ and $R_1 >> R$ such that for $|x| > R_1$

$$|\frac{1}{2\pi} \int_{B_R} \frac{-\log|x-y| + \log(1+|y|)}{\log|x|} \sum_j a_{ij}|y|^{\beta_j} e^{u_j} - m_i| \le \epsilon.$$

Also

$$|\int_{\mathbb{R}^2 \setminus B_R} \frac{-\log|x-y| + \log(1+|y|)}{\log|x|} \sum_j a_{ij}|y|^{\beta_j} e^{u_j} dy| \le \epsilon.$$

Thus $u_i - w_i \le C \log(1+|x|)$, which leads to

$$(6.2) \qquad\qquad\qquad u_i = w_i + C_i$$

for some $C_i \in \mathbb{R}$. Next we claim that

$$(6.3) \qquad\qquad m_i - \beta_i > 2 \quad \text{for all } i \in I.$$

Indeed, if this is not the case, there exists $i_0 \in I$ such that

$$m_{i_0} - \beta_{i_0} = \min\{m_1 - \beta_1, .., m_n - \beta_n\} \le 2.$$

By (6.1) and (6.2) we have

$$u_{i_0}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (-\log|x-y| + \log(1+|y|)) \sum_j a_{i_0 j}|y|^{\beta_j} e^{u_j(y)} dy - C.$$

Easy to check

$$-\log|x-y| + \log(1+|y|) \ge -\log(1+|x|),$$

thus

$$u_{i_0}(x) \ge -m_{i_0} \log(|x|+1) - C \ge -(2+\beta_{i_0}) \log(|x|+1) - C$$

a contradiction to $\int_{\mathbb{R}^2} |x|^{\beta_{i_0}} e^{u_{i_0}(x)} < \infty$. (6.3) is established. Now $u_i(x)$ can be written as

$$(6.4) \qquad u_i(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \sum_j a_{ij} |y|^{\beta_j} e^{u_j(y)} dy + c_i.$$

c_i can be determined as in the statement. Finally we derive the error term $O(r^{-\delta})$. To see this we set

$$E_1 = \{y; \quad |y| \le |x|/2.\} \quad E_2 = \{y; \quad |y - x| \le \frac{|x|}{2}.\} \quad E_3 = \mathbb{R}^2 \setminus (E_1 \cup E_2).$$

Using $e^{u_i(y)} = O(|y|^{-m_i})$ in (6.4) one obtains

$$-\frac{1}{2\pi} \int_{E_1} \log |x - y| \sum_j a_{ij} |y|^{\beta_j} e^{u_j(y)} dy = -m_i \log |x| + O(|x|^{-\delta}).$$

Similarly by elementary estimates

$$\int_{E_2 \cup E_3} \log |x - y| \sum_j a_{ij} |y|^{\beta_j} e^{u_j(y)} dy = O(|x|^{-\delta}).$$

The gradient estimate for u_i is obtained by standard estimates. Lemma 2.1 is established. $\square$

Proof of Lemma 2.2: Let $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ be defined as

$$\psi_i(t) = \phi_i(e^t), \quad i \in I.$$

Then ψ satisfies

$$\psi_i''(t) + \sum_j a_{ij} e^{u_j(e^t) + (2+\beta_j)t} \psi_j(t) = 0, \quad -\infty < t < \infty, \quad i \in I.$$

Let $\psi_{n+1} = \psi_1', ..., \psi_{2n} = \psi_n'$ and $\mathbf{F} = (\psi_1, .., \psi_{2n})^T$, then $\mathbf{F}$ satisfies

$$\mathbf{F}' = \mathbf{M}\mathbf{F}$$

where $\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$. $\mathbf{B}$ is a $n \times n$ matrix with $\mathbf{B}_{ij} = -a_{ij} e^{u_j(e^t) + (2+\beta_j)t}$. For $t > 1$, the solution for $\mathbf{F}$ is

$$(6.5) \qquad \mathbf{F}(t) = \lim_{N \to \infty} e^{\epsilon \mathbf{M}(t_N)} ... e^{\epsilon \mathbf{M}(t_0)} \mathbf{F}(0).$$

where $t_0, ..., t_N$ satisfy $t_j = j\epsilon$, $j = 0, .., N$, $\epsilon = t/N$. Since $u_i(e^t) + (2+\beta_i)t \sim (-m_i + 2 + \beta_i)t$ when t is large and $m_i > 2 + \beta_i$, we have $\|\mathbf{B}\| \sim e^{-\delta t}$ for some $\delta > 0$ and t large. With this property we further have

$$(6.6) \qquad \|\mathbf{M}\|^k \le Ce^{-k\delta_1 t}, \quad k = 2, 3, ... \quad t > 0$$

for some $\delta_1 > 0$. Using (6.6) in (6.5) we have

$$\|\mathbf{F}(t)\| = O(t), \quad t > 1.$$

Lemma 2.2 is established $\square$

Proof of Lemma 2.3: If a solution $u = (u_1, ..., u_n)$ exists, it would satisfy

$$u_i(r) = c_i(0) - \sum_{j=1}^{n} a_{ij} \int_0^r t^{\beta_j+1}(\log r - \log t)e^{u_j(t)}dt, \quad i = 1, ..., n.$$

We first prove the existence of a solution on $0 < r < \delta$ for some small $\delta > 0$ by iteration: Let $u^{(0)} = (0, .., 0)$ and

$$u_i^{(k+1)}(r) = c_i(0) - \sum_{j=1}^{n} a_{ij} \int_0^r t^{\beta_j+1}(\log r - \log t)e^{u_j^{(k)}(t)}dt, \quad i = 1, ..., n.$$

For $\delta > 0$ small and $r \in (0, \delta)$, since $\beta_i + 1 > -1$, it is easy to see that such a sequence converges. Therefore the existence of a solution for over $(0, \delta)$ is proved. The existence for $r \in (\delta, \infty)$ clearly holds because of the right hand side is a Lipschitz function of u. The proof of the uniqueness of the solution is the same as that in Lemma 2.5 later in the section. Lemma 2.3 is established. $\square$

Proof of Lemma 2.4: By Lemma 2.3 a solution to (2.2) exists for $r > 0$. We just need to show that

$$\int_0^\infty e^{u_i(r)}r^{\beta_i+1}dr < \infty, \quad \forall i \in I.$$

Let $v_i(t) = u_i(e^t) + (2 + \beta_i)t$ $(i \in I)$, then $v = (v_1, ..., v_n)$ satisfies

$$v_i''(t) + \sum_j a_{ij}e^{v_j(t)} = 0, \quad -\infty < t < \infty, \quad i \in I.$$

From the equation for u we have

$$ru_i'(r) = -\int_0^r \sum_j a_{ij}e^{u_j(s)}s^{\beta_j+1}ds < 0, \quad r > 0, \quad i \in I.$$

The last inequality is strict because $a_{ij} \geq 0$ and not all equal to 0. Consequently $v_i'(t) < 2 + \beta_i$ for $t \in \mathbb{R}$. Fix $t_0 \in \mathbb{R}$ we have, for $t > t_0$,

$$v_i'(t) = v_i'(t_0) - \int_{t_0}^t \sum_j a_{ij}e^{v_j(s)}ds \leq v_i'(t_0) - a_{ii}\int_{t_0}^t e^{v_i(s)}ds, \quad i \in I.$$

Since $a_{ii} > 0$ there exists $t > t_0$ such that $v_i'(t) < 0$. Choose $v_i'(t_1) = -\delta < 0$ for some $\delta > 0$, then we see

$$v_i(t) \leq v_i(t_1) - \delta(t - t_1), \quad t > t_1$$

which is equivalent to $u_i(r) < (-2 - \beta_i - \delta)\log r + C$ for $r > e^{t_1}$. Therefore $\int_0^\infty e^{u_i(r)}r^{\beta_i+1}dr < \infty$. Lemma 2.4 is established. $\square$

Proof of Lemma 2.5: The proof is standard, we include it for the convenience of the reader. Clearly we only need to show that $\phi_i \equiv 0$ in $(0, \delta)$ for

$\delta > 0$ small. Write $\phi_i(r)$ as

$$
\begin{aligned}
\phi_i(r) &= -\int_0^r \frac{1}{s} \int_0^s (\sum_{j \in I} a_{ij} t^{\beta_j+1} e^{u_j(t)} \phi_j(t) dt) ds \\
&= -\int_0^r \sum_{j \in I} a_{ij} t^{\beta_j+1} e^{u_j(t)} \phi_j(t) (\log r - \log t) dt
\end{aligned}
$$

Let $\alpha = \min\{\beta_1, ..., \beta_n\} + 1$, since all $\beta_i > -2$ we have $\alpha - \epsilon > -1$ for some $\epsilon > 0$ small. For the ϵ we choose $\delta > 0$ small so that $\log r - \log t < t^{-\epsilon}$ for $r < \delta$ and $t \leq r$. Thus

$$
|\phi_i(r)| \leq C \int_0^r t^{\alpha-\epsilon} \sum_j |\phi_j(t)| dt, \quad r < \delta
$$

for some C. Let $\phi(r) = \sum_{i \in I} |\phi_i(r)|$ and $F(r) = \int_0^r t^{\alpha-\epsilon} |\phi(t)| dt$, then

$$
F'(r) - Cr^{\alpha-\epsilon} F(r) \leq 0, \quad F \geq 0, \quad F(0) = 0.
$$

Since $\alpha - \epsilon > -1$, $F \equiv 0$. Lemma 2.5 is established. $\square$

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